- No book, notes, mobile phones or laptop are allowed in the exam. You can use calculator if needed.
- This exam has **two** problems:
  - Problem 1 has six parts, and is worth 25 points.
  - Problem 2 has three parts, and is worth 15 points.
- The total number of points (perfect score) is 40.

Problem 1) We have a wireless communication channel, in which the transmitter can send two symbols $x_1[n]$ and $x_2[n]$ at each time instance n, and the received signal is

$$y[n] = h_1 x_1[n] + h_2 x_2[n] + z[n],$$

where $z[n]$ is an additive white Gaussian noise, with $z[n] \sim \mathcal{N}(\mu = 0, \sigma^2 = 100)$. We can send symbols $x_i[n] \in \{-1, +1\}$ for $i = 1, 2$ over this channel.

(a) Assume the transmitter uses the same symbol for both x_1 and x_2, i.e., $x_1[n] = x_2[n]$ (we refer to this scheme as scheme-1). Find the signal-to-noise ratio (SNR) for detecting s at the receiver. *[3 points]*

(b) Now, consider scheme-2 as follows: in order to communicate two symbols s_1 and s_2, the transmitter sends

$$\begin{aligned} x_1[1] &= s_1, \qquad & x_2[1] &= s_2, \\ x_1[2] &= s_2, \qquad & x_2[2] &= -s_1, \end{aligned}$$

in two time instances. Combine $y[1]$ and $y[2]$ so that s_2 gets eliminated, and find s_1. *[5 points]*

(c) What is the SNR of this detection rule for decoding s_1? *[4 points]*

(d) Repeat parts (b) and (c) for s_2. *[2 points]*

(e) Which one of scheme-1 or scheme-2 do you recommend in order to get a better SNR? (your answer may depend on values of h_1 and h_2) *[4 points]*

(f) Now assume h_1 and h_2 are two independent random variables, each distributed as

$$P_{h_1}(x) = P_{h_2}(x) = \begin{cases} 0.2 & x = -10 \\ 0.3 & x = -1 \\ 0.3 & x = 1 \\ 0.2 & x = 10 \\ 0 & \text{otherwise.} \end{cases}$$

Hence SNR will be also a random variable for both schemes. Find the probability that the (random) SNR is at least 1 (SNR ≥ 1) for each of scheme-1 and scheme-2. *[7 points]*

Solution:

(a) By using the same symbol for both $x_1[n]$ and $x_2[n]$, we have

$$y[n] = h_1 s + h_2 s + z[n] = (h_1 + h_2)s + z[n].$$

The signal power is

$$\mathbb{E}[((h_1 + h_2)s)^2] = (h_1 + h_2)^2 \mathbb{E}[s^2] = (h_1 + h_2)^2,$$

while the noise power is $\mathbb{E}[(z[n])^2] = 100$. Hence, SNR of scheme-1 is given by

$$\mathsf{SNR}_1 = \frac{\mathbb{E}[((h_1 + h_2)s)^2]}{\mathbb{E}[(z[n])^2]} = \frac{(h_1 + h_2)^2}{100}.$$

(b) In scheme-2 we have

$$\begin{aligned} y[1] &= h_1 s_1 + h_2 s_2 + z[1] \\ y[2] &= h_1 s_2 - h_2 s_1 + z[2] \end{aligned}$$

In order to eliminate s_2, we can multiply $y[1]$ and $y[2]$ by h_1 and $-h_2$, respectively, and add them up:

$$\begin{aligned} \tilde{y}_1 &= h_1 y[1] - h_2 y[2] = h_1(h_1 s_1 + h_2 s_2 + z[1]) - h_2(h_1 s_2 - h_2 s_1 + z[2]) \\ &= (h_1^2 + h_2^2)s_1 + (h_1 z[1] - h_2 z[2]) \end{aligned}$$

(c) The resulting SNR for scheme-2 is given by

$$\text{signal power} = \mathbb{E}[((h_1^2 + h_2^2)s_1)^2] = (h_1^2 + h_2^2)^2 \mathbb{E}[s_1^2] = (h_1^2 + h_2^2)^2,$$

and

$$\text{noise power} = \mathbb{E}[(h_1 z[1] - h_2 z[2])^2] = h_1^2 \mathbb{E}[(z[1])^2] + h_2^2 \mathbb{E}[(z[2])^2] - 2h_1 h_2 \mathbb{E}[z[1]z[2]] = 100(h_1^2 + h_2^2).$$

Hence,

$$\mathsf{SNR}_2(s_1) = \frac{\text{signal power}}{\text{noise power}} = \frac{(h_1^2 + h_2^2)^2}{100(h_1^2 + h_2^2)} = \frac{h_1^2 + h_2^2}{100}.$$

(d) Decoding s_2 from $y[1]$ and $y[2]$ is similar to s_1. We can first eliminate s_1 by computing

$$\begin{aligned} \tilde{y}_2 &= h_2 y[1] + h_1 y[2] = h_2(h_1 s_1 + h_2 s_2 + z[1]) + h_1(h_1 s_2 - h_2 s_1 + z[2]) \\ &= (h_2^2 + h_1^2)s_2 + (h_2 z[1] + h_1 z[2]). \end{aligned}$$

The signal power is the same as that of s_1. For the noise power we have

$$\text{noise power} = \mathbb{E}[(h_2 z[1] + h_1 z[2])^2] = h_2^2 \mathbb{E}[(z[1])^2] + h_1^2 \mathbb{E}[(z[2])^2] + 2h_1 h_2 \mathbb{E}[z[1]z[2]] = 100(h_1^2 + h_2^2).$$

Therefore,

$$\mathsf{SNR}_2(s_2) = \frac{\text{signal power}}{\text{noise power}} = \frac{(h_1^2 + h_2^2)^2}{100(h_1^2 + h_2^2)} = \frac{h_1^2 + h_2^2}{100},$$

which is the same as $\mathsf{SNR}_2(s_1)$.

(e) Scheme-1 is better than scheme-2 only if $\mathsf{SNR}_1 \geq \mathsf{SNR}_2$, that is

$$(h_1 + h_2)^2 \geq h_1^2 + h_2^2,$$

which holds if and only if $h_1 h_2 \geq 0$, which means h_1 and h_2 have the same sign.

(f) For scheme-1 we have $\mathsf{SNR}_1 = (h_1 + h_2)^2/100$. Hence a transmitted symbol would be detected only if $(h_1 + h_2)^2 \geq 100$. From distribution of h_1 and h_2, we can see that

$h_1 + h_2$	-20	-11	-9	-2	0	2	9	11	20
prob.	0.04	0.12	0.12	0.09	0.26	0.09	0.12	0.12	0.04

and hence,

$(h_1 + h_2)^2$	0	4	81	121	400
prob.	0.26	0.18	0.24	0.24	0.08

Therefore, we have

$$\mathbb{P}(\mathsf{SNR}_1 \geq 1) = \mathbb{P}((h_1 + h_2)^2 \geq 100) = 0.24 + 0.08 = 0.32$$

On the other hand, for scheme-2, the symbol can be successfully detected only if $\mathsf{SNR}_2 = (h_1^2 + h_2^2)/100 \geq 1$, which is equivalent to $h_1^2 + h_2^2 \geq 100$. The probability distribution of h_1^2 and h_2^2 is given by

h_1^2	1	100
prob.	0.6	0.4

h_2^2	1	100
prob.	0.6	0.4

Hence,

$h_1^2 + h_2^2$	2	101	200
prob.	0.36	0.48	0.16

Therefore,

$$\mathbb{P}(\mathsf{SNR}_2 \geq 1) = \mathbb{P}(h_1^2 + h_2^2 \geq 100) = 0.48 + 0.16 = 0.64$$

One can see that probability of successful detection in scheme-2 is much higher than that of scheme-1.

Problem 2) Consider a discrete communication channel modeled as

$$y[n] = h[n]x[n]$$

where $x[n] \in \{-1, +1\}$ is the channel input, and $h[n]$ is the random channel gain with

$$h[n] = \begin{cases} 0 & \text{with probability } 0.2 \\ 1 & \text{with probability } 0.8 \end{cases}$$

Moreover, $h[n]$ and $h[n']$ are independent from each other for $n \neq n'$. We say a transmit symbol $x[n]$ is missed whenever the corresponding channel gain is zero ($h[n] = 0$).

We need to communicate an integer number m from the set $\mathcal{M} = \{0, 1, 2, \ldots, 255\}$, and we are only allowed to use the channel for ten times. To this end, we first map m to its 8-digit binary expansion $s_1 s_2 s_3 \cdots s_8$, and then send $s_1, \ldots, s_8$ along with

$$s_o = s_1 \oplus s_3 \oplus s_5 \oplus s_7$$
$$s_e = s_2 \oplus s_4 \oplus s_6 \oplus s_8$$

over the channel in ten time slots, using the mapping $0 \mapsto +1$ and $1 \mapsto -1$. For example if $m = 97$ with binary representation 01100001, we first find $s_o = 0 \oplus 1 \oplus 0 \oplus 0 = 1$ and $s_e = 1 \oplus 0 \oplus 0 \oplus 1 = 0$, and then transmit symbols $+1, -1, -1, +1, +1, +1, +1, -1, -1, +1$ over the channel corresponding to the bnary sequence 0110000110

(a) How many of the transmitted bits will be missed on average? *[2 points]*

The receiver observes $Y = (y[1], y[2], \ldots, y[10])$, from which he wants to decode m. We say m can be decoded if we can determine all the bits in its binary expansion, i.e. $s_1, s_2, \ldots, s_8$. Otherwise the decoding process fails.

(b) When does this integer decoding process fail (find conditions under which we cannot find all the 8 desired bits)? *[7 points]*

(c) Find the probability of failure in the decoding process. *[6 points]*

Solution:

(a) Note that $y[n] = x[n]$ if $h[n] = 1$. Otherwise, $y[n] = 0$ regardless of $x[n]$ if $h[n] = 0$. Hence, each bit will be received (not missed) if $h[n] = 1$, which happens with probability 0.8.

Let N be the number of channel uses in which $h[n] = 1$. We have

$$\mathbb{E}[N] = \mathbb{E}\left[\sum_{n=1}^{10} \mathbb{1}_{\{h[n]=1\}}\right] = \sum_{n=1}^{10} \mathbb{E}[\mathbb{1}_{\{h[n]=1\}}] = \sum_{n=1}^{10} \mathbb{P}[h[n] = 1] = \sum_{n=1}^{10} 0.8 = 8.$$

(b) If none of the transmitted are missed then we can clearly decode m. Similarly, if only one of the transmitted symbols is missing we can still recover that by the help of s_o and s_e.

If three or more symbols are missing, then we definitely cannot recover $s_1, \ldots, s_8$.

If exactly two of the symbols are missing, then we may of may not be able to recover m. Let s_i and s_j be missing. We can distinguish the following cases:

$$\begin{aligned} i, j \in \{1, 3, 5, 7, o\} &\Rightarrow \text{ the missing bits cannot be recovered;} \\ i, j \in \{2, 4, 6, 8, e\} &\Rightarrow \text{ the missing bits cannot be recovered;} \\ i \in \{1, 3, 5, 7, o\} \text{ and } j \in \{2, 4, 6, 8, e\} &\Rightarrow \text{ the missing bits can be recovered;} \end{aligned}$$

(c) From part (b) we have

$$\begin{aligned} \mathbb{P}[\text{success}] &= \mathbb{P}[\text{no missing symbols}] + \mathbb{P}[\text{1 missing symbols}] + \mathbb{P}[i \in \{1, 3, 5, 7, o\} \text{ and } j \in \{2, 4, 6, 8, e\}] \\ &= \binom{10}{0}(0.2)^0(0.8)^{10} + \binom{10}{1}(0.2)^1(0.8)^9 + \left(\binom{5}{1}(0.2)^1(0.8)^4\right)\left(\binom{5}{1}(0.2)^1(0.8)^4\right) \\ &= (0.8)^{10} + 10(0.2)(0.8)^9 + 25(0.2)^2(0.8)^8 = 0.5436 \end{aligned}$$